MULTIPLICITY OF POSITIVE SOLUTIONS FOR A FRACTIONAL LAPLACIAN EQUATIONS INVOLVING CRITICAL NONLINEARITY

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Abstract. In this paper we deal with the multiplicity of positive solutions to the fractional Laplacian equation

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \lambda f(x)|u|^{q-2}u + |u|^{2^*_\alpha - 2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary, $0 < \alpha < 2$, $(-\Delta)^{\frac{\alpha}{2}}$ stands for the fractional Laplacian operator, $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ may be sign changing and λ is a positive parameter. We will prove that there exists $\lambda_* > 0$ such that the problem has at least two positive solutions for each $\lambda \in (0, \lambda_*)$. In addition, the concentration behavior of the solutions are investigated.

1. Introduction

In this paper, we are concerned with the multiplicity of solutions to the following fractional Laplacian equation:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \lambda f(x)|u|^{q-2}u + |u|^{2^*_\alpha - 2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary, $0 < \alpha < 2$, $(-\Delta)^{\frac{\alpha}{2}}$ stands for the fractional Laplacian operator, $2^*_\alpha := \frac{2N}{N-\alpha}$, $1 < q < 2$, $\lambda > 0$ and $f : \Omega \to \mathbb{R}$ is a continuous function with $f^+(x) = \max\{f(x), 0\} \neq 0$ on Ω. From the assumptions on f and q, we know that the problem (1.1) involving the concave-convex nonlinearities and sign-changing weight function.

The fractional power of the Laplacian is the infinitesimal generators of Lévy stable diffusion process and arises in anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics and American options in finance. For more details, one can see [1, 9] and references therein.

Recently the fractional Laplacian attracts much interest in nonlinear analysis, such as in [2, 3, 5, 6, 7, 12, 13, 14, 16]. Caffarelli and Silvestre [5] gave a new formulation of the fractional Laplacians through Dirichlet-Neumann maps. This is commonly used in the recent literature since it allows us to write nonlocal problems in a local way and this permits to us use the variational methods for those

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kinds of problems. In [7], Cabré and Tan defined the operator of the square root of Laplacian through the spectral decomposition of the Laplacian operator on Ω with zero Dirichlet boundary conditions. With classical local techniques, they established existence of positive solutions for problems with subcritical nonlinearities, regularity and L^∞-estimate of Brezis-Kato type for weak solutions. In [12, 13, 16], the authors employed the Brezis-Nirenberg technique to build an analogue results to the problem in [4], but with the fractional Laplacian instead of the Laplacian.

The analogue problem to problem (1.1) for the Laplacian operator has been investigated widely in the past decades, see for example [8, 15, 17] and the references therein. The main purpose of this paper is to generalize the partial results of [2] to the problem involving sign-changing weight function. Using the variational methods and the Nehari manifold decomposition, we first prove that the problem (1.1) has at least two positive solutions for λ sufficiently small.

Theorem 1.1. *There exists $\lambda_* > 0$ such that for $\lambda \in (0\,,\,\lambda_*)$, the problem (1.1) has at least two positive solutions.*

As for the asymptotic behavior of the solutions obtained in Theorem 1.1 as $\lambda \to 0$, we have the following result.

Theorem 1.2. *Assume that a sequence $\{\lambda_n\}$ satisfies $\lambda_n > 0$ and*

$$\lambda_n \to 0 \ \text{as}\ n \to \infty.$$

Then there exists a subsequence $\{\lambda_n\}$ and two sequence $\{u_n^{(j)}(x)\}$ $(\,j = 1.2\,)$ of positive solutions of Eq.(1.1) such that

(i) $\|u_n^{(1)}\|_{H_0^{\frac{\alpha}{2}}(\Omega)} \to 0$ *as* $n \to \infty$;

(ii) There exist two sequence $\{x_n\} \subset \Omega$, $\{R_n\} \subset \mathbb{R}^+$ and a positive solution $u_0 \in H_0^{\frac{\alpha}{2}}(\Omega)$ of critical problem

$$(-\Delta)^{\frac{\alpha}{2}}u = |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^N,$$

such that

$$R_n \to +\infty \ \text{as}\ n \to +\infty$$

and

$$\|u_n^{(2)}(x) - R_n^{\frac{N-\alpha}{2}} u_0(R_n(x - x_n))\|_{H_0^{\frac{\alpha}{2}}(\Omega)} \to 0 \ \text{as}\ n \to \infty.$$

The paper is organized as follows. In Section 2, we introduce a variational setting of the problem and present some preliminary results. In Section 3, some properties of the fractional operator are discussed. Then we give the proof of Theorem 1.1. Finally, the proof of Theorem 1.2 is given in Section 4.

For convenience we fix some notations. $L^p(\Omega)$ $(1 < p \le \infty)$ denotes the usual Sobolev space with norm $|\cdot|_{L^p}$; $C_0(\bar{\Omega})$ denotes the space of continuous real functions in $\bar{\Omega}$ vanishing on the boundary $\partial\Omega$; C or $C_i (i = 1, 2, \cdots,)$ denote any positive constant.

2. Notation and Preliminaries

Denote the upper half-space in $\mathbb{R}_+^{N+1}$ by

$$\mathbb{R}_+^{N+1} = \{z = (x, y) = (x_1, x_2, \cdots, x_n, y) \in \mathbb{R}^{N+1} |\, y > 0\},$$

the half cylinder standing on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ by $\mathcal{C}_\Omega = \Omega \times (0\,,\infty) \subset \mathbb{R}^{N+1}_+$ and its lateral boundary given that $\partial_L \mathcal{C}_\Omega = \partial\Omega \times [0\,,\infty)$.

Let $\{\varphi_j\}$ be an orthonormal basis of $L^2(\Omega)$ with $|\varphi_j|_{L^2} = 1$ forming a spectral decomposition of $-\Delta$ in Ω with zero Dirichlet boundary conditions and λ_j be the corresponding eigenvalues. Let

$$H_0^{\frac{\alpha}{2}}(\Omega) = \{u = \sum_{j=1}^{\infty} a_j\varphi_j \in L^2(\Omega) : \|u\|_{H_0^{\frac{\alpha}{2}}(\Omega)} = \Big(\sum_{j=1}^{\infty} a_j^2 \lambda^{\frac{\alpha}{2}}\Big)^{\frac{1}{2}} < \infty\}.$$

Define the inner product in $H_0^{\frac{\alpha}{2}}(\Omega)$ by

$$\langle u\,,\,v\rangle_{H_0^{\frac{\alpha}{2}}(\Omega)} = \int_\Omega (-\Delta)^{\frac{\alpha}{4}} u (-\Delta)^{\frac{\alpha}{4}} v\,dx.$$

It is not difficult to see that $H_0^{\frac{\alpha}{2}}(\Omega)$ is a Hilbert space. For any $u \in H_0^{\frac{\alpha}{2}}(\Omega)$, $u = \sum\limits_{j=1}^{\infty} a_j\varphi_j$ with $a_j = \int_\Omega u\,\varphi_j dx$, the fractional power of the Dirichlet Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is defined by

$$(-\Delta)^{\frac{\alpha}{2}} u = \sum_{j=1}^{\infty} a_j\,\lambda_j^{\frac{\alpha}{2}}\varphi_j.$$

Definition 2.1. *We say that $u \in H_0^{\frac{\alpha}{2}}(\Omega)$ is a solution of Eq.* (1.1) *such that for every function $\varphi \in H_0^{\frac{\alpha}{2}}(\Omega)$, it holds*

$$\int_\Omega (-\Delta)^{\frac{\alpha}{4}} u (-\Delta)^{\frac{\alpha}{4}} \varphi dx = \lambda \int_\Omega f(x)|u|^{q-2}u\varphi dx + \int_\Omega |u|^{2^*_\alpha - 2} u\varphi dx.$$

Associated with problem (1.1) we consider the energy functional

$$I(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx - \frac{\lambda}{q}\int_\Omega f(x)|u|^q dx - \frac{1}{2^*_\alpha}\int_\Omega |u|^{2^*_\alpha} dx.$$

This functional is well defined in $H_0^{\frac{\alpha}{2}}(\Omega)$, and moreover, the critical points of I correspond to weak solutions of problem (1.1).

We now conclude the main ingredients of a recently developed technique which can deal with fractional power of the Laplacian. To treat the nonlocal problem (1.1), we will study a corresponding extension problem, so that we can investigate problem (1.1) by studying a local problem via classical nonlinear variational methods.

We first define the extension operator and fractional Laplacian for functions in $H_0^{\frac{\alpha}{2}}(\Omega)$.

Definition 2.2. *Given a function $u \in H_0^{\frac{\alpha}{2}}(\Omega)$, we define its α-harmonic extension $w = E_\alpha(u)$ to the cylinder $\mathcal{C}_\Omega$ as a solution of the problem*

$$\begin{cases} div(y^{1-\alpha}\nabla w) = 0, & \text{in } \mathcal{C}_\Omega,\\ w = 0, & \text{on } \partial_L\mathcal{C}_\Omega,\\ w = u, & \text{on } \Omega\times\{0\}.\end{cases}$$

Following [5], we can define the fractional Laplacian operator by the Dirichlet to Neumann map as follows.

Definition 2.3. *For any regular function $u(x)$, the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ acting on u is defined by*

$$(-\Delta)^{\frac{\alpha}{2}}u(x) = -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y), \quad \forall x\in\Omega, \quad y\in(0\,,\,\infty),$$

where $w = E_\alpha(u)$ and κ_α is a normalization constant.

Define $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} = \Big(\kappa_\alpha\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy\Big)^{\frac{1}{2}}.$$

From [3] and [5], the map $E_\alpha(\cdot)$ is an isometry between $H_0^{\frac{\alpha}{2}}(\Omega)$ and $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$. Furthermore, we have

(i) $\|(-\Delta)^{\frac{\alpha}{2}}u\|_{H^{-\frac{\alpha}{2}}(\Omega)} = \|u\|_{H_0^{\frac{\alpha}{2}}(\Omega)} = \|E_\alpha(u)\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}$, where $H^{-\frac{\alpha}{2}}(\Omega)$ denotes the dual space of $H_0^{\frac{\alpha}{2}}(\Omega)$;

(ii) For any $w\in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$, there exists a constant C independent of w such that

$$\|\mathrm{tr}_\Omega w\|_{L^r(\Omega)} \le C\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}$$

holds for every $r\in[2\,,\,\frac{2N}{N-\alpha}]$. Moreover, $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$ is compactly embedded into $L^r(\Omega)$ for $r\in[2\,,\,\frac{2N}{N-\alpha})$.

Now we can transform the nonlocal problem (1.1) into the following local problem:

$$\begin{cases} -\,div(y^{1-\alpha}\nabla w) = 0, & \text{in } \mathcal{C}_\Omega,\\ w = 0, & \text{on } \partial_L\mathcal{C}_\Omega,\\ \dfrac{\partial w}{\partial \upsilon^\alpha} = \lambda f(x)|w|^{q-2}w + |w|^{2^*_\alpha-2}w & \text{on } \Omega\times\{0\}, \end{cases} \tag{2.1}$$

where $\frac{\partial w}{\partial\upsilon^\alpha} := -\kappa_\alpha\lim\limits_{y\to0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x,y)$, $\forall x\in\Omega$. An energy solution to this problem is a function $w\in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$ such that

$$\kappa_\alpha\int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}\langle\nabla w\,,\,\nabla\varphi\rangle dxdy = \lambda\int\limits_{\Omega\times\{0\}} f(x)|w|^{q-2}w\,\varphi dx + \int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha-2}w\,\varphi dx$$

for all $\varphi\in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$.

If w satisfies (2.1), then the trace $u = \mathrm{tr}_\Omega w = w(x,0)\in H_0^{\frac{\alpha}{2}}(\Omega)$ is an energy solution to problem (1.1). The converse is also true. By the equivalence of these two formulations, we will use both formulations in the sequel to their best advantage.

The associated energy functional to problem (2.1) is

$$J(w) = \frac{\kappa_\alpha}{2}\int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2dxdy - \frac{\lambda}{q}\int\limits_{\Omega\times\{0\}} f(x)|w|^q dx - \frac{1}{2^*_\alpha}\int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha}dx,$$

for all $w\in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$. Clearly, the critical points of J in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$ correspond to critical points of I in $H_0^{\frac{\alpha}{2}}(\Omega)$.

In the following lemma, we will list some inequalities.

Lemma 2.1. *For every* $1 \le r \le \frac{2N}{N-\alpha}$, *and every* $w \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$, *it holds*

$$\Big(\int_{\Omega\times\{0\}} |w|^r dx\Big)^{\frac{2}{r}} \le C\,\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy,$$

where the constant C *depends on* r, α, N, $|\Omega|$.

Lemma 2.2. *For every* $w \in H^{\frac{\alpha}{2}}(\mathbb{R}^{N+1}_+)$, *it holds*

$$S(\alpha,N)\Big(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-\alpha}}\Big)^{\frac{N-\alpha}{N}} \le \int_{\mathbb{R}^{N+1}_+} y^{1-\alpha}|\nabla w|^2 dxdy, \tag{2.2}$$

where $u = tr_\Omega\, w$. *The best constant takes the exact value*

$$S(\alpha,N) = \frac{2\,\pi^{\frac{\alpha}{2}}\Gamma(\frac{2-\alpha}{2})\Gamma(\frac{N+\alpha}{2})(\Gamma(\frac{N}{2}))^{\frac{\alpha}{N}}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{N-\alpha}{2})(\Gamma(N))^{\frac{\alpha}{N}}}$$

and can be achieved when $u(x) = w(x,0)$ *takes the form*

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-\alpha}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-\alpha}{2}}} \tag{2.3}$$

for $\varepsilon > 0$ *arbitrary and* $w = E_\alpha(u)$.

Now we are looking for the solutions of problem (1.1). Equivalently, we consider the solutions of problem (2.1). First we consider the Nehari minimization problem, i.e., for $\lambda > 0$,

$$m_J = \inf\{J(w)|\, w \in \mathcal{N}\},$$

where

$$\mathcal{N} = \{w \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega) \mid \langle J'(w)\,,\, w\rangle = 0\}.$$

Define

$$\Psi(w) = \langle J'(w)\,,\, w\rangle = \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \lambda \int_{\Omega\times\{0\}} f(x)|w|^q dx - \int_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx.$$

Then, for $w \in \mathcal{N}$,

$$\langle \Psi'(w)\,,\, w\rangle = 2\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \lambda\, q \int_{\Omega\times\{0\}} f(x)|w|^q dx - 2^*_\alpha \int_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx.$$

Similar to the method used in [17, 18], we split $\mathcal{N}$ into three parts:

$$\mathcal{N}^+ = \{w \in \mathcal{N} \mid \langle \Psi'(w)\,,\, w\rangle > 0\};$$
$$\mathcal{N}^0 = \{w \in \mathcal{N} \mid \langle \Psi'(w)\,,\, w\rangle = 0\};$$
$$\mathcal{N}^- = \{w \in \mathcal{N} \mid \langle \Psi'(w)\,,\, w\rangle < 0\}.$$

Then we have the following results.

Lemma 2.3. *Let* $\lambda_1 = \Big(\frac{2^*_\alpha-2}{2^*_\alpha-q}\Big)\Big(\frac{2-q}{2^*_\alpha-q}\Big)^{\frac{2-q}{2^*_\alpha-q}}\Big(\kappa_\alpha S(\alpha,N)\Big)^{\frac{2^*_\alpha-q}{2^*_\alpha-2}}|f|^{-1}_\infty$. *Then for every* $w \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$, $w \ne 0$ *and* $\lambda \in (0\,,\,\lambda_1)$, *there exist unique* $t^+(w)$ *and* $t^-(w)$ *such that*

(1) $0 \leq t^+(w) < t_{max} = \Big(\frac{(2-q)\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy}{(2^*_\alpha - q) \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx}\Big)^{\frac{1}{2^*_\alpha - 2}} < t^-(w)$;

(2) $t^-(w)w \in \mathcal{N}^-$ *and* $t^+(w)w \in \mathcal{N}^+$;

(3) $\mathcal{N}^- = \Big\{ w \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega) \setminus \{0\} : t^-(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}) = \|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \Big\}$;

(4) $J(t^- w) = \max\limits_{t \geq t_{max}} J(tw)$ *and* $J(t^+ w) = \min\limits_{t\in[0,t^-]} J(tw)$.

Moreover, $t^+(w) > 0$ *if and only if* $\int\limits_{\Omega\times\{0\}} f(x)|w|^q dx > 0$.

Proof. The proof is almost the same as that in [17]. We need only to define

$$s(t) = t^{2-q}\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - t^{2^*_\alpha - q} \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx.$$

Thus, we omit the details here. □

Lemma 2.4. *There exists* $\lambda_2 > 0$ *such that for each* $\lambda \in (0\,,\, \lambda_2)$*, we have* $\mathcal{N}^0 = \{0\}$.

Proof. Suppose the contrary, there exists a $w \in \mathcal{N}^0 \setminus \{0\}$, such that

$$\langle \Psi'(w)\,,\, w\rangle = 0. \tag{2.4}$$

Then, we consider the following two cases.

Case (i): $\int\limits_{\Omega\times\{0\}} f(x)|w|^q dx = 0$. Then

$$\begin{aligned}\langle \Psi'(w)\,,\, w\rangle &= 2\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \lambda\, q \int\limits_{\Omega\times\{0\}} f(x)|w|^q dx - \frac{2N}{N-\alpha} \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx \\ &= 2\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \frac{2N}{N-\alpha}\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy \\ &= -\frac{2\alpha}{N-\alpha}\|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} < 0.\end{aligned}$$

So, in this case $w \in \mathcal{N}^-$.

Case (ii): $\int\limits_{\Omega\times\{0\}} f(x)|w|^q dx \neq 0$. From (2.4), we get that

$$\begin{aligned} 0 &= 2\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \lambda\, q \int\limits_{\Omega\times\{0\}} f(x)|w|^q dx - 2^*_\alpha \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx \\ &= (2-q)\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - (2^*_\alpha - q) \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx.\end{aligned}$$

This implies that

$$\|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} = \frac{2^*_\alpha - q}{2-q} \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx. \tag{2.5}$$

Moreover, we have

$$\begin{aligned}\lambda \int_{\Omega\times\{0\}} f(x)|w|^q dx &= \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \int_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx \\ &= \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \frac{2-q}{2^*_\alpha - q}\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy \qquad (2.6)\\ &= \frac{2^*_\alpha - 2}{2^*_\alpha - q}\|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}.\end{aligned}$$

Then, by (2.6), (2.2) and the Hölder inequality, we obtain

$$\|w\|^{2-q}_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \le \lambda\Big(\frac{2^*_\alpha - q}{2^*_\alpha - 2}\Big)\Big(\kappa_\alpha S(\alpha,N)\Big)^{-\frac{q}{2}}|f|_{L^\infty}. \qquad (2.7)$$

Let $K : \mathcal{N} \to \mathbb{R}$ be given by

$$K(w) = C(N,\alpha)\|w\|^{\frac{2(2^*_\alpha-1)}{2^*_\alpha-2}}_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\Big(\int_{\Omega\times\{0\}} |w|^{2^*_\alpha}dx\Big)^{\frac{1}{2-2^*_\alpha}} - \lambda \int_{\Omega\times\{0\}} f(x)|w|^q dx,$$

where $C(N,\alpha) = \Big(\frac{2^*_\alpha-2}{2-q}\Big)\Big(\frac{2-q}{2^*_\alpha-q}\Big)^{\frac{2^*_\alpha-1}{2^*_\alpha-2}}$. Then $K(w) = 0$ for all $w \in \mathcal{N}^0$. Indeed, by (2.5) and (2.6),

$$\begin{aligned}K(w) &= C(N,\alpha)\|w\|^{\frac{2(2^*_\alpha-1)}{2^*_\alpha-2}}_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\Big(\int_{\Omega\times\{0\}} |w|^{2^*_\alpha}dx\Big)^{\frac{1}{2-2^*_\alpha}} - \lambda \int_{\Omega\times\{0\}} f(x)|w|^q dx \\ &= \Big(\frac{2^*_\alpha - 2}{2-q}\Big)\Big(\frac{2-q}{2^*_\alpha - q}\Big)^{\frac{2^*_\alpha-1}{2^*_\alpha-2}}\Big(\frac{2^*_\alpha - q}{2-q}\Big)^{\frac{1}{2^*-2}}\|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \lambda \int_{\Omega\times\{0\}} f(x)|w|^q dx \\ &= \frac{2^*_\alpha - 2}{2^*_\alpha - q}\|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \lambda\int_{\Omega\times\{0\}} f(x)|w|^q dx \\ &= 0.\end{aligned}$$

On the other hand, by (2.6),(2.7), we have

$$\begin{aligned}K(w) &\ge C(N,\alpha)\|w\|^{\frac{2(2^*_\alpha-1)}{2^*_\alpha-2}}_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\Big(\int_{\Omega\times\{0\}} |w|^{2^*_\alpha}dx\Big)^{\frac{1}{2-2^*_\alpha}} - \lambda|f|_{L^\infty}|w|^q_{L^{2^*_\alpha}} \\ &\ge C(N,\alpha)\Big(\kappa_\alpha S(\alpha,N)\Big)^{\frac{N+\alpha}{2\alpha}}\Big(\int_{\Omega\times\{0\}} |w|^{2^*_\alpha}dx\Big)^{\frac{1}{2^*_\alpha}} - \lambda|f|_{L^\infty}|w|^q_{L^{2^*_\alpha}} \qquad (2.8)\\ &\ge \lambda|f|_{L^\infty}|w|^q_{L^{2^*}}\Big[\frac{\Big(\frac{2^*_\alpha-2}{2^*_\alpha-q}\Big)^{\frac{1}{2-q}}\Big(2-q\Big)^{\frac{n-\alpha}{2\alpha}}\Big(\kappa_\alpha S(\alpha,N)\Big)^{\frac{q-1}{2-q}+\frac{N+\alpha}{2\alpha}}}{(\lambda|f|_{L^\infty})^{\frac{1}{2-q}}} - 1\Big].\end{aligned}$$

This implies that there exists

$$\lambda_2 := |f|^{-1}_{L^\infty}\Big(\frac{2^*_\alpha - 2}{2^*_\alpha - q}\Big)\Big(2-q\Big)^{\frac{(N-\alpha)(2-q)}{2\alpha}}(\kappa_\alpha S(\alpha,N))^{\frac{2\alpha(q-1)+(N+\alpha)(2-q)}{2\alpha}}$$

such that for each $\lambda \in (0\,,\,\lambda_2)$, we have $K(w) > 0$ for all $w \in \mathcal{N}^0 \setminus \{0\}$, which yields a contradiction. Thus, we can conclude that $\mathcal{N}^0 = \{0\}$ for all $\lambda \in (0\,,\,\lambda_2)$. ☐

Lemma 2.5. *If $w \in \mathcal{N}^+$ and $w \neq 0$, then* $\int\limits_{\Omega\times\{0\}} f(x)|w|^q dx > 0$.

Proof. From $w \in \mathcal{N}^+$, we have

$$\begin{aligned}
2\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy &> \lambda q \int\limits_{\Omega\times\{0\}} f(x)|w|^q dx + 2^*_\alpha \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx \\
&= q\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy + (2^*_\alpha - q) \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx,
\end{aligned}$$

that is,

$$\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy > \frac{2^*_\alpha - q}{2-q} \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx.$$

Then, we have

$$\begin{aligned}
\lambda \int\limits_{\Omega\times\{0\}} f(x)|u|^{q+1} dx &= \kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx \\
&> \frac{2^*_\alpha - 2}{2-q} \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha} dx > 0.
\end{aligned}$$

This completes the proof. ☐

The following lemma shows that the minimizers on $\mathcal{N}$ are actually the critical points of J.

Lemma 2.6. *For $\lambda \in (0\,,\,\lambda_2)$. If $w \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$ is a local minimizer for J on $\mathcal{N}$, then $J'(w) = 0$ in $H^{-\frac{\alpha}{2}}(\mathcal{C}_\Omega)$, where $H^{-\frac{\alpha}{2}}(\mathcal{C}_\Omega)$ denotes the dual space of $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$.*

Proof. If $w_0 \in \mathcal{N}$ is a local minimizer of J, then w_0 is a nontrivial solution of the optimization problem

$$\text{minimize } J(w) \text{ subject to } \langle \Psi'(w) = 0,$$

Hence by the theory of Lagrange multiplies, there exists $\theta \in \mathbb{R}$ such that $J'(w_0) = \theta\Psi'(w_0)$ in $H^{-\frac{\alpha}{2}}(\mathcal{C}_\Omega)$. This implies that

$$\langle J'(w_0)\,,\,w_0\rangle = \theta\langle \Psi'(w_0)\,,\,w_0\rangle. \tag{2.9}$$

By Lemma 2.4, for every $w \neq 0$, we have $\langle \Psi'(w_0)\,,\,w_0\rangle \neq 0$ and so by (2.9), $\theta = 0$. This completes the proof. ☐

Lemma 2.7. *The functional J is coercive and bounded from below on $\mathcal{N}$.*

Proof. For $w \in \mathcal{N}$, we have

$$\begin{aligned} J(w) &= (\frac{1}{2}-\frac{1}{2^*_\alpha})\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \lambda(\frac{1}{q}-\frac{1}{2^*_\alpha}) \int\limits_{\Omega\times\{0\}} f(x)|u|^q dx \\ &\geq \frac{\alpha}{2N}\|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \lambda\Big(\frac{2^*_\alpha - q}{q2^*_\alpha}\Big)|f|_{L^\infty}\Big(\kappa_\alpha\, S(\alpha,N)\Big)^{-\frac{q}{2}}\|w\|^q_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \\ &\geq \frac{q-2}{2}\Big(\frac{N}{\alpha}\Big)^{\frac{q}{2-q}}\Big(\lambda\, C\Big)^{\frac{2}{2-q}}, \end{aligned}$$

where $C = \Big(\frac{2^*_\alpha - q}{q2^*_\alpha}\Big)|f|_{L^\infty}\kappa_\alpha^{-\frac{q}{2}}\, S(\alpha,N)^{-\frac{q}{2}}$. This tell us that J is coercive and bounded from below on $\mathcal{N}$. □

In the end of this section, we will use the idea of [15] to get the property of $\mathcal{N}$.

Lemma 2.8. *For each $w \in \mathcal{N}$, $w \not\equiv 0$, there exists $r > 0$ and a differentiable function $t = t(v)$ such that $t = t(v) > 0$ for all $v \in \{w \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega) : \|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} < \varepsilon\}$ satisfying that*

$$t(0) = 1,\ t(v)(w-v) \in \mathcal{N},$$

and

$$\langle t'(0)\,,\,v\rangle = \frac{2\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}\nabla w \nabla v dxdy - q\lambda \int\limits_{\Omega\times\{0\}} f(x)|w|^{q-2}wvdx - 2^*_\alpha \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha-2}wvdx}{(2-q)\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - (2^*_\alpha - q)\int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha}dx}$$

for all $v \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$.

Proof. Define $F : \mathbb{R}\times H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega) \to \mathbb{R}$ as follows.

$$\begin{aligned} F(t,v) &= \langle J'(t(w-v))\,,\,t(w-v)\rangle \\ &= t^2\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla(w-v)|^2 dxdy - \lambda\, t^q \int\limits_{\Omega\times\{0\}} f(x)|w-v|^q dx - t^{2^*_\alpha}\int\limits_{\Omega\times\{0\}} |w-v|^{2^*_\alpha}dx, \end{aligned}$$

for all $v \in \mathcal{N}$.

Since $F(1,0) = \langle J'(w)\,,\,w\rangle = 0$ and by Lemma 2.4, we obtain

$$\begin{aligned} F'_t(1,0) &= 2\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \lambda q \int\limits_{\Omega\times\{0\}} f(x)|w|^q dx - 2^*_\alpha \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha}dx \\ &= (2-q)\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - (2^*_\alpha - q)\int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha}dx \neq 0. \end{aligned}$$

Applying the implicit function theorem at the point $(1,0)$, we get that there exist $\varepsilon > 0$ small and a function $t = t(v)$ satisfying $t(0) = 1$ and

$$\langle t'(0)\,,\,v\rangle = \frac{2\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}\nabla w \nabla v dxdy - q\lambda \int\limits_{\Omega\times\{0\}} f(x)|w|^{q-2}wvdx - 2^*_\alpha \int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha-2}wvdx}{(2-q)\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - (2^*_\alpha - q)\int\limits_{\Omega\times\{0\}} |w|^{2^*_\alpha dx}}.$$

Moreover, there is a $t(v)$ such that $F(t(v), v) = 0$ for all $v \in \{w \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega) : \|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} < \varepsilon\}$, which is equivalent to $\langle J'(t(v)(w-v)), t(v)(w-v)\rangle = 0$, that is, $t(v)(w-v) \in \mathcal{N}$. We prove the lemma. □

3. Proof of Theorem 1.1

3.1. The minimizer solution on $\mathcal{N}^+$. Let

$$\lambda_* = \min\{\lambda_1 , \lambda_2\}. \tag{3.1}$$

In this subsection, we show that problem (2.1) has a position solution if $\lambda < \lambda_*$, which is the minimizer of J on $\mathcal{N}^+$.

By Lemma 2.7, for $\lambda \in (0 , \lambda_*)$, J is coercive and bounded from below on $\mathcal{N}$ and so on $\mathcal{N}^+$. Therefore, we define

$$m^+ = \inf\{J(w) : w \in \mathcal{N}^+\}.$$

Now we consider the following auxiliary equation:

$$\begin{cases} -div(y^{1-\alpha}\nabla w) = 0, & \text{in } \mathcal{C}_\Omega, \\ w = 0, & \text{on } \partial_L\mathcal{C}_\Omega, \\ \dfrac{\partial w}{\partial v^\alpha} = \lambda f(x)|w|^{q-2}w & \text{on } \Omega \times \{0\}, \end{cases} \tag{3.2}$$

In this case we use the notation F and $\mathcal{M}$ respectively, for the energy functional and the natural constrain, namely,

$$F(w) = \frac{\kappa_\alpha}{2} \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \frac{\lambda}{q} \int_{\Omega\times\{0\}} f(x)|w|^q dx.$$

$$\mathcal{M} = \{w \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega) \setminus \{0\} : \langle F'(w) , w\rangle = 0\}.$$

Setting

$$m_F = \inf\{F(w) : w \in \mathcal{M}\},$$

then we have the following result.

Theorem 3.1. *For each $\lambda > 0$, problem (3.2) has a positive solution w_0 such that*

$$F(w_0) = m_\lambda < 0.$$

Proof. We start by showing that F is coercive, bounded from below on $\mathcal{M}$ and $m_\lambda < 0$. Indeed, for any $w \in \mathcal{M}$, we have

$$\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|w|^2 dxdy = \lambda \int_{\Omega\times\{0\}} f(x)|w|^q dx \le \lambda|f|_{L^\infty} (\kappa_\alpha S(\alpha, N)^{-\frac{q}{2}})\|w\|^q_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}. \tag{3.3}$$

This implies

$$F(w) \ge \frac{1}{2}\|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \frac{1}{q}\lambda\,|f|_{L^\infty} \Big(\kappa_\alpha S(\alpha, N)\Big)^{-\frac{q}{2}} \|w\|^q_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)},$$

and therefore, we easily derive the coerciveness for $1 < q < 2$. Moreover, (3.3) implies

$$\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \le \Big(\lambda|f|_{L^\infty} (\kappa_\alpha S(\alpha, N))^{-\frac{q}{2}}\Big)^{\frac{1}{2-q}}. \tag{3.4}$$

Hence, for all $w \in \mathcal{M}$ we have

$$F(w) = (\frac{1}{2} - \frac{1}{q})\|w\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \geq -\frac{2-q}{2q}\Big(\lambda |f|_{L^s}\,(\kappa_\alpha S(\alpha,N))^{-\frac{q}{2}}\Big)^{\frac{2}{2-q}}.$$

So F is bounded from below on $\mathcal{M}$ and $m_\lambda < 0$.

Let $\{w_n\}_n$ be a minimizing sequence of F on $\mathcal{M}$. Then, by (3.4) and the compact imbedding theorem, there exists a subsequence of $\{w_n\}_n$, still denoted by $\{w_n\}_n$, and w_0 such that

$$\begin{aligned} & w_n \rightharpoonup w_0 \text{ weakly in } H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega); \\ & w_n(\cdot,0) \to w_0(\cdot,0) \text{ strongly in } L^p(\Omega) \text{ for } 1<p<2^*_\alpha; \\ & w_n(\cdot,0) \to w_0(\cdot,0) \text{ a.e. in } \Omega. \end{aligned} \tag{3.5}$$

Now, we claim that $\int_{\Omega\times\{0\}} f(x)|w_0|^q dx > 0$. If not, by (3.5) we obtain

$$\int_{\Omega\times\{0\}} f(x)|w_0|^q dx = 0$$

and

$$\int_{\Omega\times\{0\}} f(x)|w_n|^q dx \to 0 \text{ as } n\to\infty.$$

Hence $\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w_n|^2 dxdy \to 0$ and $F(w_n) \to 0$ as $n\to\infty$ which is contradicts $F(w_n) \to m_\lambda < 0$ as $n\to\infty$. Therefore, we have $\int_{\Omega\times\{0\}} f(x)|w_0|^q dx > 0$. In particular $w_0 \not\equiv 0$.

Next, we prove $w_n \to w_0$ $(n\to\infty)$ strongly in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$. Let us suppose on the contrary that

$$\|w_0\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} < \liminf_{n\to\infty} \|w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \quad \text{as } n\to\infty$$

and

$$\int_{\Omega\times\{0\}} f(x)|w_n|^q dx \to \int_{\Omega\times\{0\}} f(x)|w_0|^q dx \quad \text{as } n\to\infty.$$

So

$$\|w_0\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \lambda\int_{\Omega\times\{0\}} f(x)|w_0|^q dx < \liminf_{n\to\infty}\Big(\|w_n\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \lambda\int_{\Omega\times\{0\}} f(x)|w_n|^q dx\Big) = 0. \tag{3.6}$$

From $\int_{\Omega\times\{0\}} f(x)|w_0|^q dx > 0$ and (3.6), we known that the function

$$F(t\,w_0) = \frac{t^2}{2}\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla v_0|^2 dxdy - \frac{\lambda\, t^q}{q}\int_{\Omega\times\{0\}} f(x)|w_0|^q dx$$

is initially decreasing and eventually increasing on t with a single turning point $t_0 \neq 1$ such that $t_0 w_0 \in \mathcal{M}$. Then from $t_0 w_n \rightharpoonup t_0 w_0$ and (3.6) we get that

$$F(t_0 w_0) < F(w_0) < \liminf_{n\to\infty} F(w_n) = m_\lambda$$

which is a contradiction. Hence $w_n \to w_0$ strongly in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$. This implies $w_0 \in \mathcal{M}$ and $F(w_0) = m_\lambda$. Moreover, it follows from $F(w_0) = F(|w_0|)$ and $|w_0| \in \mathcal{M}$ that w_0 is a nonnegative weak solution to (3.2). Then by the strong maximum principle [11] we have $w_0 > 0$ in $\mathcal{C}_\Omega$, that is, w_0 is a positive solution of problem (3.2). □

Now, we establish the existence of a minimum for J on $\mathcal{N}^+$.

Proposition 3.1. *For each $\lambda \in (0\,,\,\lambda_*)$, the functional J has a minimizer w_1 in $\mathcal{N}$.*

Proof. From Lemma 2.7, it is easily derived the coerciveness and the lower boundedness of J on $\mathcal{N}$. Clearly, by the Ekeland's variational principle applying for the minimization problem $\inf_{\mathcal{N}} J(w)$, there exists a minimizing sequence $\{w_n\} \subset \mathcal{N}$ such that

$$J(w_n) < m_J + \frac{1}{n}, \tag{3.7}$$

and

$$J(Z) \geq J(w_n) - \frac{1}{n}\|w_n - Z\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}, \quad \forall\, Z \in \mathcal{N}. \tag{3.8}$$

Let w_0 be a positive solution of (3.2) satisfying $F(w_0) = m_F < 0$. Then

$$\begin{aligned} m_\lambda = F(w_0) &= \frac{\kappa_\alpha}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|w_0|^2 dxdy - \frac{\lambda}{q}\int_{\Omega\times\{0\}} f(x)|w_0|^q dx \\ &= (\frac{1}{2} - \frac{1}{q})\|w_0\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}, \end{aligned}$$

that is,

$$\|w_0\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} = \frac{2q}{q-2} m_\lambda > 0. \tag{3.9}$$

By Lemma 2.5 in [17], for w_0, there exists a positive constant t_1 such that $t_1 w_0 \in \mathcal{N}^+$, i.e.,

$$\int_{\Omega\times\{0\}} |t_1 w_0|^{2^*_\alpha} dx < \frac{2-q}{2^*_\alpha - q}\,\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla(t_1 w_0)|^2 dxdy. \tag{3.10}$$

Then, from (3.9) and (3.10),

$$\begin{aligned} J(t_1\, w_0) &= \frac{q-2}{2q}\,\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla(t_1 w_0)|^2 dxdy + \frac{2^*_\alpha - q}{q\, 2^*_\alpha}\int_{\Omega\times\{0\}} |t_1 w_0|^{2^*_\alpha} dx \\ &< \Big(\frac{1}{2} - \frac{1}{2^*_\alpha}\Big)\frac{q-2}{q}\|t_1 w_0\|^2 = \frac{\alpha}{2N}\,\frac{q-2}{q} t_1^2\, \frac{2q}{q-2} m_\lambda \\ &= \frac{\alpha}{N} t_1^2\, m_\lambda < 0. \end{aligned}$$

This yields

$$m_J \leq m^+ < 0. \tag{3.11}$$

So (3.7), (3.11) and the coerciveness of J imply that the minimizer sequence $\{w_n\}$ is bounded, and so there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$, and

w_1 such that

$$\begin{aligned}
&w_n \rightharpoonup w_1 \text{ weakly in } H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega);\\
&w_n(\cdot,0) \to w_1(\cdot,0) \text{ strongly in } L^p(\Omega) \text{ for } 1 \le p < 2^*_\alpha;\\
&w_n(\cdot,0) \to w_1(\cdot,0) \text{ a.e. in } \Omega.
\end{aligned}$$

Now, we claim that $w_1 \not\equiv 0$. In fact, suppose on the contrary that $w_1 \equiv 0$. Since $w_n \in \mathcal{N}$, we deduce

$$\begin{aligned}
J(w_n) &= \frac{\kappa_\alpha}{2}\int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w_n|^2 dxdy - \frac{\lambda}{q}\int\limits_{\Omega\times\{0\}} f(x)|w_n|^q dx - \frac{1}{2^*_\alpha}\int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx\\
&= \frac{2\alpha}{N}\|w_n\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \lambda\,\frac{2^*_\alpha - q}{2^*_\alpha q}\int\limits_{\Omega\times\{0\}} f(x)|w_n|^q dx\\
&> -\lambda\,\frac{2^*_\alpha - q}{q2^*_\alpha}\int\limits_{\Omega\times\{0\}} f(x)|w_n|^q dx,
\end{aligned}$$

that is,

$$\int\limits_{\Omega\times\{0\}} f(x)|w_n|^q dx > -\frac{q2^*_\alpha}{\lambda(2^*_\alpha - q)}J(w_n) \ge -\frac{q2^*_\alpha}{\lambda(2^*_\alpha - q)}m_J > 0,$$

which clearly shows that $w_1 \not\equiv 0$.

Next, we will show that

$$\|J'(w_n)\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \to 0 \text{ as } n \to \infty.$$

Exactly the same as in Lemma 2.8 we may apply suitable function $t_n : B_\varepsilon(0) \to \mathbb{R}^+$ for some $\varepsilon > 0$ small such that

$$t_n(v)(w_n - v) \in \mathcal{N},\ \forall\, v \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega),\ \|v\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} < \varepsilon.$$

Set $\eta_n = t_n(v)(w_n - v)$. Since $\eta_n \in \mathcal{N}$, we deduce from (3.8) that

$$J(\eta_n) - J(w_n) \ge -\frac{1}{n}\|\eta_n - w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}.$$

By the mean value theorem, we have

$$\langle J'(w_n)\,,\, \eta_n - w_n\rangle \ge -\frac{1}{n}\|\eta_n - w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} + o(\|\eta_n - w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}). \tag{3.12}$$

Thus, from $\eta_n - w_n = (t_n(v) - 1)(w_n - v) - v$ and (3.12), we get

$$\begin{aligned}
&\langle J'(w_n)\,,\, -v\rangle + (t_n(v) - 1)\langle J'(w_n)\,,\, w_n - v\rangle\\
&\qquad \ge -\frac{1}{n}\|\eta_n - w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} + o(\|\eta_n - w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}).
\end{aligned} \tag{3.13}$$

Let $v = \frac{r\, w_1}{\|w_1\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}$, $0 < r < \varepsilon$. Substituting into (3.13), we have

$$\langle J'(w_n)\,,\, \frac{w_1}{\|w_1\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\rangle \leq \frac{1}{n\,r}\,\|\eta_n - w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} + \frac{1}{r}\,o(\|\eta_n - w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}) + \frac{(t_n(v)-1)}{r}\langle J'(w_n) - J'(\eta_n)\,,\, w_n - v\rangle. \tag{3.14}$$

Since

$$\begin{aligned}\|\eta_n - w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} &= \|(t_n(v)-1)\, w_n - t_n(v)\, v\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \\ &\leq \varepsilon|t_n(v)| + |t_n(v)-1|\|w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\end{aligned} \tag{3.15}$$

and

$$\lim_{r\to 0}\frac{|t_n(v)-1|}{r} = \lim_{r\to 0}\frac{|\langle t_n'(0)\,,\, v\rangle|}{r} \leq \|t_n'(0)\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}. \tag{3.16}$$

If we let $r \to 0$ in the right hand of (3.14) for a fixed n, then by (3.15), (3.16) and the boundedness of w_n, we can find a constant $C > 0$ such that

$$\langle J'(w_n)\,,\, \frac{w_1}{\|w_1\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\rangle \leq \frac{C}{n}\Big(1 + \|t_n'(0)\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\Big). \tag{3.17}$$

We are done once we show that $\|t_n'(0)\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}$ is uniformly bounded in n. Since

$$\langle t_n'(0)\,,\, \varphi\rangle = \frac{2\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\nabla v_n \nabla\varphi dxdy - q\lambda \int_{\Omega\times\{0\}} f(x)|w_n|^{q-2}w_n\varphi dx - 2^*_\alpha \int_{\Omega\times\{0\}} |w|^{2^*_\alpha-2}w_n\varphi dx}{(2-q)\|w_n\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - (2^*_\alpha - q)\int_{\Omega\times\{0\}}|w_n|^{2^*_\alpha}dx},$$

we have by the boundness of w_n,

$$\|t_n'(0)\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \leq \frac{C_1}{|(2-q)\|w_n\|^2 - (2^*_\alpha - q)\int_{\Omega\times\{0\}}|w_n|^{2^*_\alpha}dx|}, \tag{3.18}$$

for some suitable positive constant C_1. We next only need to show that

$$\Big|(2-q)\|w_n\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - (2^*_\alpha - q)\int_{\Omega\times\{0\}}|w_n|^{2^*_\alpha}dx\Big| \geq c > 0 \tag{3.19}$$

for some $c > 0$ and n large enough. Arguing by contradiction, assume that there exists a subsequence $\{w_n\}$ such that

$$(2-q)\|w_n\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - (2^*_\alpha - q)\int_{\Omega\times\{0\}}|w_n|^{2^*_\alpha}dx \to 0 \text{ as } n\to\infty. \tag{3.20}$$

Then,

$$\begin{aligned}\lim_{n\to\infty}\int_{\Omega\times\{0\}}|w_n|^{2^*_\alpha}dx &= \lim_{n\to\infty}\frac{2-q}{2^*_\alpha - q}\|w_n\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \\ &\geq \frac{2-q}{2^*_\alpha - q}\|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} > 0.\end{aligned} \tag{3.21}$$

Therefore, we can find a constant $C_2 > 0$ such that

$$\int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx > C_2 \tag{3.22}$$

for n large enough. In addition, (3.20) and the fact that $w_n \in \mathcal{N}^+$ also give as

$$\lambda \int\limits_{\Omega\times\{0\}} f(x)|w_n|^q dx = \|u_n\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx = \frac{2\alpha}{(N-\alpha)(2-q)} \int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx + o(1)$$

and

$$\|w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \leq \Big[\lambda \frac{(2^*_\alpha - q)(N-\alpha)}{2\,\alpha} |f|_{L^\infty} S^{-\frac{N}{N-\alpha}}\Big]^{\frac{1}{2-q}} + o(1). \tag{3.23}$$

This implies $K(w_n) = o(1)$, where K is given in Section 2.

However, by (3.22), (3.23), similar to the calculation of (2.8), for each $\lambda \in (0\,,\,\lambda_*)$, there is a $C_3 > 0$ such that

$$K(w_n) > C_3,$$

which is impossible.

Hence, from (3.17), (3.18) and (3.19),

$$\langle J'(w_n)\,,\, \frac{w_1}{\|w_1\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\rangle \leq \frac{C}{n}$$

for some $C > 0$. Taking $n \to \infty$, we get $\|J'(w_n)\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \to 0$. This shows that $\{w_n\}$ is a (PS) sequence of functional J.

Finally, we prove that $w_n \to w_1$ strongly in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$. Since $w_n \rightharpoonup w_1$ weakly in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$, it follows that

$$\begin{aligned} m_J \leq J(w_1) &= \frac{1}{2}\|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \frac{\lambda}{q} \int\limits_{\Omega\times\{0\}} f(x)|w_1|^q dx - \frac{1}{2^*_\alpha} \int\limits_{\Omega\times\{0\}} |w_1|^{2^*_\alpha} dx \\ &= \frac{\alpha}{2N}\|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \lambda\,\frac{2^*_\alpha - q}{q2^*_\alpha} \int\limits_{\Omega\times\{0\}} f(x)|w_1|^q dx \\ &\leq \lim_{n\to\infty} J(w_n) = m_J. \end{aligned}$$

Consequently, $w_n \to w_1$ strongly in $H^{\frac{\alpha}{2}}_0(\Omega)$ and $J(w_1) = m_J$. The proof is completed. □

Theorem 3.2. *For each $\lambda \in (0\,,\,\lambda_*)$, the problem* (1.1) *admits a positive solution in $\mathcal{N}^+$.*

Proof. From Proposition 3.1, we have that w_1 is a nontrivial solution of problem (2.1). Moreover, we have

$$w_1 \in \mathcal{N}^+.$$

In fact, if $w_1 \in \mathcal{N}^-$, by Lemma 2.3, there exists a unique $t^-(w_1) > 0$, $t^+(w_1) > 0$ such that $t^-(w_1)\,w_1 \in \mathcal{N}^-$, then we have $t^-(w_1) = 1$ and $t^+(w_1) < 1$. Since $J(t^+(w_1)\,w_1) = \min\limits_{t\in[0,t^-(w_1)]} J(t\,w_1)$, we can find a $t_0 \in (t^+(w_1)\,,\,t^-(w_1))$ such that

$$J(t^+(w_1)\,w_1) < J(t_0\,w_1) \leq J(t^-(w_1)\,w_1) = J(1\cdot w_1) = m_J,$$

which implies that $w_1 \in \mathcal{N}^+$. Since $J(w_1) = J(|w_1|)$ and $|w_1| \in \mathcal{N}^+$, we can take $w_1 \geq 0$. By the strong maximum principle [11], we get $w_1 > 0$ in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$. Hence, $u_1(x) = w_1(x, 0) \in H_0^{\frac{\alpha}{2}}(\Omega)$ is a positive solution of problem (1.1) and $J(w_1) = m^+$. We complete the proof. $\square$

Remark 3.1. *For $w_1 \in \mathcal{N}^+$, by the Hölder inequality and the Young inequality we have*

$$\begin{aligned}
0 > J(w_1) &= \frac{\alpha}{2N} \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w_1|^2 dxdy - \lambda \frac{2^*_\alpha - q}{q\, 2^*_\alpha} \int_{\Omega\times\{0\}} f(x)|w|^q dx \\
&\geq \frac{\alpha}{2N} \|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \lambda \frac{2^*_\alpha - q}{q\, 2^*_\alpha} |f|_{L^\infty} \Big(\kappa_\alpha S(\alpha, N)\Big)^{-\frac{q}{2}} \|w_1\|^q_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \\
&\geq -\lambda \frac{2-q}{q\, 2^*_\alpha} \Big(|f|_{L^\infty}\, \kappa_\alpha S(\alpha, N)^{-\frac{q}{2}}\Big)^{\frac{2}{2-q}}.
\end{aligned}$$

So, we deduce that $J(w_1) \to 0$ as $\lambda \to 0$.

3.2. **The minimizer solution on $\mathcal{N}^-$.** In the following, we prove that problem (2.1) has a solution in $\mathcal{N}^-$. Since J is coercive and bounded from below on $\mathcal{N}$ and so on $\mathcal{N}^-$,

$$m^- = \inf_{\mathcal{N}^-} J(w).$$

Then, there exists a minimizing sequence $\{w_n\} \subset \mathcal{N}^-$ such that

$$J(w_n) \to m^- \quad \text{as } n \to \infty. \tag{3.24}$$

Lemma 3.1. *The set $\mathcal{N}^-$ is closed.*

Proof. Suppose that there are some $w_n \in \mathcal{N}^-$ and $w_n \to w_0 \notin \mathcal{N}^-$, then $w_0 \in \mathcal{N}^0 = \{0\}$. For $w_n \in \mathcal{N}^-$, we have

$$0 \leq (2-q)\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w_n|^2 dxdy < (2^*_\alpha - q) \int_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx \to 0.$$

This implies that $\lim\limits_{n\to\infty} \kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w_n|^2 dxdy = 0$. Note that if $w_n \in \mathcal{N}^-$, then $\|w_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}$ $\geq \gamma > 0$ for a suitable $\gamma > 0$. This is a contradiction. Hence we have $w_0 \in \mathcal{N}^-$, and so $\mathcal{N}^-$ is closed. $\square$

Next, we will use the trace inequality (2.2) to the family of minimizers $w_\varepsilon = E_\alpha(u_\varepsilon)$, where u_ε is given in (2.3).

Note that f is a indefinite continuous function on Ω and $f^+ \not\equiv 0$, where $f^+ = \max\{f(x)\,,\, 0\}$, then the set $\Sigma := \{x \in \Omega :\ f(x) > 0\} \subset \Omega$ is an open set with positive measure. Without loss of generality, we may assume that Σ is a domain.

Let $\eta \in C_0^\infty(\mathcal{C}_\Sigma)$, $0 \leq \eta \leq 1(\ \forall (x, y) \in \Sigma \times (0, \infty))$, be a positive function satisfying

$$\Big(\mathrm{supp} f^+ \times \{y > 0\}\Big) \cap \{(x, y) \in \mathcal{C}_\Sigma :\ \eta = 1\} \neq \emptyset.$$

Moreover, for small fixed $\rho > 0$,

$$\eta(x,y) = \begin{cases} 1, & (x,y) \in B_\rho, \\ 0, & (x,y) \notin \overline{B_{2\rho}}, \end{cases}$$

where $B_\rho = \{(x,y) : |(x,y)| < \rho, \, y > 0, \, x \in \Sigma\}$. We take ρ small enough such that $\overline{B_{2\rho}} \subset \overline{\mathcal{C}_\Sigma}$. Note that $\eta\, w_\varepsilon \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$.

Let $\lambda_* > 0$ be as in (3.1). Then for $\lambda \in (0\,,\,\lambda_*)$ we have the following result.

Lemma 3.2. *Let w_1 be the local minimum in Proposition 3.1. Then for $\varepsilon > 0$ small enough,*

$$\sup_{t \geq 0} J(w_1 + t\eta w_\varepsilon) < m_J + \frac{\alpha}{2N}\Big(\kappa_\alpha S(\alpha, N)\Big)^{\frac{N}{\alpha}}.$$

Proof. First, we have

$$\begin{aligned} J(w_1 + t\eta w_\varepsilon) &= \frac{\kappa_\alpha}{2} \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla(w_1 + t\eta w_\varepsilon)|^2 dxdy - \frac{\lambda}{q} \int_{\Omega \times \{0\}} f(x)|w_1 + t\eta w_\varepsilon|^q dx \\ &\quad - \frac{1}{2^*_\alpha} \int_{\Omega \times \{0\}} |w_1 + t\eta w_\varepsilon|^{2^*_\alpha} dx \\ &= \frac{1}{2}\|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} + \frac{t^2}{2}\|\eta w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} + t\langle w_1\,,\,\eta w_\varepsilon\rangle \\ &\quad - \frac{\lambda}{q} \int_{\Omega \times \{0\}} f(x)|w_1 + t\eta w_\varepsilon|^q dx - \frac{1}{2^*_\alpha} \int_{\Omega \times \{0\}} |w_1 + t\eta w_\varepsilon|^{2^*_\alpha} dx. \end{aligned} \tag{3.25}$$

Since w_1 is a solution of Eq.(2.1), we get that

$$\frac{1}{2}\|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} = J(w_1) + \frac{\lambda}{q} \int_{\Omega \times \{0\}} f(x)|w_1|^q dx + \frac{1}{2^*_\alpha} \int_{\Omega \times \{0\}} |w_1|^{2^*_\alpha} dx; \tag{3.26}$$

$$t\langle w_1\,,\,\eta w_\varepsilon\rangle = t\lambda \int_{\Omega \times \{0\}} f(x)|w_1|^{q-1} \eta w_\varepsilon dx + t \int_{\Omega \times \{0\}} |w_1|^{2^*_\alpha - 1} \eta w_\varepsilon dx; \tag{3.27}$$

$$\begin{aligned} \int_{\Omega \times \{0\}} |w_1 + t\eta w_\varepsilon|^{2^*_\alpha} dx &= \int_{\Omega \times \{0\}} |w_1|^{2^*_\alpha} dx + t^{2^*_\alpha} \int_{\Omega \times \{0\}} |\eta w_\varepsilon|^{2^*_\alpha} dx + 2^*_\alpha t \int_{\Omega \times \{0\}} |w_1|^{2^*_\alpha - 2} w_1 \eta w_\varepsilon dx \\ &\quad + 2^*_\alpha t^{2^*_\alpha - 1} \int_{\Omega \times \{0\}} |\eta w_\varepsilon|^{2^*_\alpha - 2} \eta w_\varepsilon w_1 dx + o(\varepsilon^{\frac{N-\alpha}{2}}); \end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
&\int_{\Sigma\times\{0\}} f(x)\Big(|w_1+t\eta w_\varepsilon|^q-|w_1|^q+qt|w_1|^{q-1}\eta w_\varepsilon\Big)dx\\
&= q\int_{\Sigma\times\{0\}} f^+(x)\Big\{\int_0^{t\eta w_\varepsilon}(|w_1+\tau|^{q-1}+|w_1|^{q-1}\tau)d\tau\Big\}dx\\
&\geq q\int_{\Sigma\times\{0\}} f^+(x)\Big\{\int_0^{t\eta w_\varepsilon}(|w_1+\tau|^{q-1}+|w_1|^{q-1}\tau)d\tau\Big\}dx\\
&\geq 0.
\end{aligned}\tag{3.29}$$

Substituting (3.26),(3.27),(3.28) and (3.29) in (3.25) and using the fact that $\eta\in C_0^\infty(\mathcal{C}_\Sigma)$, we obtain

$$\begin{aligned}
&J(w_1+t\eta w_\varepsilon)\\
&= J(w_1)-\frac{\lambda}{q}\int_{\Omega\times\{0\}} f(x)(|w_1+t\eta w_\varepsilon|^q-|w_1|^q)dx+t\langle w_1\,,\,\eta w_\varepsilon\rangle-t\int_{\Omega\times\{0\}}|w_1|^{2^*_\alpha-1}\eta w_\varepsilon dx\\
&+\frac{t^2}{2}\|\eta w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}-\frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\Omega\times\{0\}}|\eta w_\varepsilon|^{2^*_\alpha}dx-t^{2^*_\alpha-1}\int_{\Omega\times\{0\}}|\eta w_\varepsilon|^{2^*_\alpha-1}w_1dx+o(\epsilon^{\frac{N-\alpha}{2}})\\
&= J(w_1)-\frac{\lambda}{q}\int_{\Sigma\times\{0\}} f(x)(|w_1+t\eta w_\varepsilon|^q-|w_1|^q+qt|w_1|^{q-1}\eta w_\varepsilon)dx\\
&+\frac{t^2}{2}\|\eta w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}-\frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\Omega\times\{0\}}|\eta w_\varepsilon|^{2^*_\alpha}dx-t^{2^*_\alpha-1}\int_{\Omega\times\{0\}}|\eta w_\varepsilon|^{2^*_\alpha-1}w_1dx+o(\epsilon^{\frac{N-\alpha}{2}})\\
&\leq J(w_1)+\frac{t^2}{2}\|\eta w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}-\frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\Omega\times\{0\}}|\eta w_\varepsilon|^{2^*_\alpha}dx\\
&-t^{2^*_\alpha-1}\int_{\Omega\times\{0\}}|\eta w_\varepsilon|^{2^*_\alpha-1}w_1dx+o(\epsilon^{\frac{N-\alpha}{2}}).
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\Omega\times\{0\}}|\eta w_\varepsilon|^{2^*_\alpha-1}dx &= \int_{\Omega\times\{0\}}\Big[\frac{\eta\varepsilon^{\frac{N-\alpha}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-\alpha}{2}}}\Big]^{\frac{N+\alpha}{N-\alpha}}dx\\
&= \int_{\mathbb{R}^N}\frac{\varepsilon^{\frac{N+\alpha}{2}}}{\varepsilon^{N+\alpha}(1+|z|^2)^{\frac{N+\alpha}{2}}}\,\varepsilon^N dz\\
&= C\,\varepsilon^{\frac{N-\alpha}{2}}\int_0^{+\infty}\frac{1}{(1+r^2)^{\frac{N+\alpha}{2}}}\\
&\leq C\,\varepsilon^{\frac{N-\alpha}{2}},
\end{aligned}$$

and from [2, 14], we have

$$\|\eta w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}=\|w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}+O(\varepsilon^{N-\alpha}),$$

$$\int_{\Omega\times\{0\}} |\eta w_\varepsilon|^{2^*_\alpha} dx = \int_{\mathbb{R}^N} \Big(\frac{\varepsilon}{\varepsilon^2+|x|^2}\Big)^N dx + O(\varepsilon^N).$$

Thus,

$$J(w_1+t\eta w_\varepsilon) \le J(w_1)+\frac{t^2}{2}\,\|w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\Omega\times\{0\}} |w_\varepsilon|^{2^*_\alpha} dx + O(\varepsilon^N) - C\,\varepsilon^{\frac{N-\alpha}{2}} + o(\epsilon^{\frac{N-\alpha}{2}}) \tag{3.30}$$

Let

$$h(t) = \frac{t^2}{2}\,\|w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\Omega\times\{0\}} |w_\varepsilon|^{2^*_\alpha} dx.$$

for $t>0$. Since $h(t)$ goes to $-\infty$ as $t\to\infty$, $\sup_{t\ge 0} h(t)$ is achieved at some $t_\varepsilon>0$ with $h'(t_\varepsilon)=0$. That is

$$0 = \|w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - t_\varepsilon^{2^*_\alpha-2}\int_{\Omega\times\{0\}} |w_\varepsilon|^{2^*_\alpha} dx.$$

Therefore,

$$h(t) \le h(t_\varepsilon) = \Big(\frac{1}{2}-\frac{1}{2^*_\alpha}\Big)\|w_\varepsilon\|^{\frac{2^*_\alpha}{2^*_\alpha-2}}_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\Big(\int_{\Omega\times\{0\}} |w_\varepsilon|^{2^*_\alpha} dx\Big)^{-\frac{2}{2^*_\alpha-2}}. \tag{3.31}$$

On the other hand, since w_ε are minimizers of the trace inequality of (2.2), we have that

$$\|w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} = \kappa_\alpha\, S(\alpha,N)\Big(\int_{\Omega\times\{0\}} |w_\varepsilon|^{2^*_\alpha} dx\Big)^{\frac{2}{2^*_\alpha}}. \tag{3.32}$$

Hence, from (3.30),(3.31) and (3.32), we obtain

$$\begin{aligned} J(w_1+t\eta w_\varepsilon) &\le J(w_1) + \frac{\alpha}{2N}\Big(\kappa_\alpha\, S(\alpha,N)\Big)^{\frac{N}{\alpha}} + O(\varepsilon^N) - C\varepsilon^{\frac{N-\alpha}{2}} + o(\varepsilon^{\frac{N-4}{2}}) \\ &< m_J + \frac{\alpha}{2N}\Big(\kappa_\alpha\, S(\alpha,N)\Big)^{\frac{N}{\alpha}}, \end{aligned}$$

for $\varepsilon>0$ sufficiently small. □

The following proposition provides a precise description of the (PS)-sequence of J.

Proposition 3.2. *If every minimizing sequence $\{u_n\}$ of J on $\mathcal{N}^-$ satisfies $m_J \le J(w_n) < m_J + \frac{\alpha}{2N}(\kappa_\alpha S)^{\frac{N}{\alpha}}$, then $\{w_n\}$ satisfies the* (PS)*-condition on $\mathcal{N}^-$.*

Proof. By (3.24) and $\{w_n\}\subset\mathcal{N}^-$, it is easy to prove that the sequence $\{w_n\}$ is bounded in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$. Them we can extract a subsequence, still denoted by $\{w_n\}$, and w_2 such that, as $n\to\infty$,

$$\begin{aligned} &w_n \rightharpoonup w_2 \quad \text{weakly in } H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega);\\ &w_n(\cdot,0)\to w_2(\cdot,0) \quad \text{strongly in } L^p(\Omega), \forall 1\le p<2^*_\alpha;\\ &w_n(\cdot,0)\to w_2(\cdot,0) \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.33}$$

Since $\{w_n\} \subset \mathcal{N}^-$ is a minimizing sequence, by the Lagrange multiplier method, we get that $J'(w_n) \to 0$ as $n \to \infty$. Consequently, by (3.33) we have

$$\langle J'(w_2)\,,\,\varphi\rangle = 0, \quad \forall \varphi \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega).$$

Then w_2 is a solution in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$ for problem (2.1), and $J(w_2) \geq m_J$.

First, we claim that $w_2 \not\equiv 0$. If not, by (3.33) we have $\int\limits_{\Omega\times\{0\}} f(x)|w_2|^q dx \to 0$ as $n \to \infty$. Thus, form $J'(w_n) \to 0$, we obtain that

$$\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w_n|^2 dxdy = \int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx + o(1). \tag{3.34}$$

and

$$\begin{aligned} J(w_n) &= \frac{\kappa_\alpha}{2}\int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w_n|^2 dxdy - \frac{\lambda}{q}\int\limits_{\Omega\times\{0\}} f(x)|w_n|^q dx - \frac{1}{2^*_\alpha}\int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx \\ &= \frac{\alpha}{2N}\int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx < m_J + \frac{\alpha}{2N}(\kappa_\alpha\, S(\alpha,N))^{\frac{N}{\alpha}} \\ &< \frac{\alpha}{2N}(\kappa_\alpha\, S(\alpha,N))^{\frac{N}{\alpha}} \text{ (since } m_J < 0). \end{aligned}$$

So, we get

$$\int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx < (\kappa_\alpha S(\alpha,N))^{\frac{N}{\alpha}}. \tag{3.35}$$

On the other hand, from (3.34) and (2.2), we have that

$$\int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx \geq (\kappa_\alpha S(\alpha,N))^{\frac{N}{\alpha}}.$$

This contradicts (3.35). Then $w_2 \not\equiv 0$ and $J(w_2) \geq m_J$.

We write $\hat{w}_n = w_n - w_2$ with $\hat{w}_n \rightharpoonup 0$ weakly in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$. By the Brezis-Lieb Lemma, we have

$$\int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx = \int\limits_{\Omega\times\{0\}} |w_n - w_2|^{2^*_\alpha} dx = \int\limits_{\Omega\times\{0\}} |w_n|^{2^*_\alpha} dx - \int\limits_{\Omega\times\{0\}} |w_2|^{2^*_\alpha} dx + o(1).$$

Hence, for n large enough, we can conclude that

$$\begin{aligned} m_J + \frac{\alpha}{2N}(\kappa_\alpha S(\alpha,N))^{\frac{N}{\alpha}} &> J(w_2 + \hat{w}_n) \\ &= J(w_2) + \frac{\kappa_\alpha}{2}\int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla \hat{w}_n|^2 dxdy - \frac{1}{2^*_\alpha}\int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx + o(1) \\ &\geq m_J + \frac{\kappa_\alpha}{2}\int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla \hat{w}_n|^2 dxdy - \frac{1}{2^*_\alpha}\int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx + o(1), \end{aligned}$$

this is,

$$\frac{\kappa_\alpha}{2}\int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla \hat{w}_n|^2 dxdy - \frac{1}{2^*_\alpha}\int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx < \frac{\alpha}{2N}(\kappa_\alpha S(\alpha,N))^{\frac{N}{\alpha}} + o(1). \tag{3.36}$$

Since $J'(w_n) \to 0$ as $n \to \infty$, $\{w_n\}$ is uniformly bounded and w_2 is a solution of Eq. (2.1), it follows

$$\begin{aligned} o(1) &= \langle J'(w_n)\,,\, w_n\rangle \\ &= J'(w_2) + \kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla \hat{w}_n|^2 dxdy - \int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx + o(1) \\ &= \kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla \hat{w}_n|^2 dxdy - \int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx + o(1). \end{aligned}$$

We obtain

$$\kappa_\alpha \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla \hat{w}_n|^2 dxdy = \int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx + o(1) \quad (n \to \infty). \tag{3.37}$$

We claim that (3.36) and (3.37) can hold simultaneously only if $\{\hat{w}_n\}$ admits a subsequence which converges strongly to zero. If not, then $\|\hat{w}_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}$ is bounded away from zero, that is $\|\hat{w}_n\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} > c > 0$. From (3.37) and (2.2) then it follows

$$\int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx \geq \Big(\kappa_\alpha S(\alpha, N)\Big)^{\frac{N}{\alpha}} + o(1). \tag{3.38}$$

By (3.36), (3.37) and (3.38), for n large enough, we have

$$\begin{aligned} \frac{\alpha}{2N}\Big(\kappa_\alpha S(\alpha, N)\Big)^{\frac{N}{4}} &\leq \frac{\alpha}{2N} \int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx + o(1) \\ &= \frac{\kappa_\alpha}{2} \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla \hat{w}_n|^2 dxdy - \frac{1}{2^*_\alpha} \int\limits_{\Omega\times\{0\}} |\hat{w}_n|^{2^*_\alpha} dx + o(1) \\ &< \frac{\alpha}{2N}\Big(\kappa_\alpha S(\alpha, N)\Big)^{\frac{N}{\alpha}}, \end{aligned}$$

which is a contradiction. Consequently, $w_n \to w_2$ strongly in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$ and $w_2 \in \mathcal{N}^-$. $\square$

Next, we establish the existence of a local minimum for J on $\mathcal{N}^-$.

Proposition 3.3. *For any $\lambda \in (0\,,\, \lambda_*)$, the functional J has a minimizer $w_2 \in \mathcal{N}^-$ such that*

$$J(w_2) = m_- < m_J + \frac{\alpha}{2N}\Big(\kappa_\alpha S(\alpha, N)\Big)^{\frac{N}{\alpha}}.$$

Proof. For every $w \in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$, by Lemma 2.3, we can find a unique $t^-(w) > 0$ such that $t^-(w)w \in \mathcal{N}^-$. Define

$$W_1 = \{w :\, w = 0 \text{ or } t^-\Big(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\Big) > \|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\},$$

$$W_2 = \{w :\, t^-\Big(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\Big) < \|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\}.$$

Then $\mathcal{N}^-$ disconnects $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$ in two connected components W_1 and W_2 and $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)\setminus\mathcal{N}^- = W_1\cup W_2$.

For each $w\in\mathcal{N}^+$, there exist unique $t^-(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}})>0$ and $t^+(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}})>0$ such that

$$t^+\Big(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\Big)<t_{max}<t^-\Big(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\Big);$$

$$t^+\Big(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\Big)\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\in\mathcal{N}^+;$$

and

$$t^-\Big(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\Big)\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\in\mathcal{N}^-.$$

Since $w\in\mathcal{N}^+$, we have $t^+(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}})\frac{1}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}=1$. By the fact that $t^+(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}})<t^-(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}})$, we get

$$t^-\Big(\frac{w}{\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\Big)>\|w\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)},$$

and then $\mathcal{N}^+\subset W_1$. In particular, $w_1\in W_1$ is the minimizer of J in $\mathcal{N}^+$.

Now, we claim that there exists $l_0>0$ such that $w_1+l_0\eta w_\varepsilon\in W_2$. First, we find a constant $c>0$ such that $0<t^-(\frac{w_1+l\eta w_\varepsilon}{\|w_1+l\eta w_\varepsilon\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}})<c$ for each $l>0$. Otherwise, there exists a sequence $\{l_n\}$ such that $l_n\to\infty$ and $t^-(\frac{w_1+l_n\eta w_\varepsilon}{\|w_1+l_n\eta w_\varepsilon\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}})\to\infty$ as $n\to\infty$. Let $\widetilde{w}_n=\frac{w_1+l_n\eta w_\varepsilon}{\|w_1+l_n\eta w_\varepsilon\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}$. By Lemma 2.3, we obtain $t^-(\widetilde{w}_n)\widetilde{w}_n\in\mathcal{N}^-$. Then we have

$$\begin{aligned}\int\limits_{\Omega\times\{0\}}|\widetilde{w}_n|^{2^*_\alpha}dx&=\frac{1}{\|w_1+l_n\eta w_\varepsilon\|^{2^*_\alpha}_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\int\limits_{\Omega\times\{0\}}|w_1+l_n\eta w_\varepsilon|^{2^*_\alpha}dx\\&=\frac{1}{\|\frac{w_1}{l_n}+\eta w_\varepsilon\|^{2^*_\alpha}_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\int\limits_{\Omega\times\{0\}}|\frac{w_1}{l_n}+\eta w_\varepsilon|^{2^*_\alpha}dx\\&\to\frac{1}{\|\eta w_\varepsilon\|^{2^*_\alpha}_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\int\limits_{\Omega\times\{0\}}|\eta w_\varepsilon|^{2^*_\alpha}dx>0(n\to\infty),\end{aligned}$$

and

$$\begin{aligned}&J(t^-(\widetilde{w}_n)\widetilde{w}_n)\\&=\frac{1}{2}[t^-(\widetilde{w}_n)]^2-\frac{\lambda}{q}[t^-(\widetilde{w}_n)]^q\int\limits_{\Omega\times\{0\}}f(x)\widetilde{w}_n^q dx-\frac{[t^-(\widetilde{w}_n)]^{2^*_\alpha}}{2^*_\alpha}\int\limits_{\Omega\times\{0\}}|\widetilde{w}_n|^{2^*_\alpha}dx\\&\to-\infty\quad(n\to\infty).\end{aligned}$$

This contradicts that J is bounded below on $\mathcal{N}$.

Let

$$l_0 = \frac{\sqrt{\left|c^2 - \|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\right|}}{\|\eta w_\varepsilon\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}} + 1.$$

Then

$$\begin{aligned}
\|w_1 + l_0\eta w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} &= \|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} + l_0^2\|\eta w_\varepsilon\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} + 2l_0\langle w_1\,,\,\eta w_\varepsilon\rangle \\
&\geq \|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} + \left|c^2 - \|w_1\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}\right| + 2l_0\langle w_1\,,\,\eta w_\varepsilon\rangle \\
&\geq c^2 \\
&> [t^-\Big(\frac{w_1 + l_0\eta w_\varepsilon}{\|w_1 + l_0\eta w_\varepsilon\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}\Big)]^2,
\end{aligned}$$

that is, $w_1 + l_0\eta w_\varepsilon \in W_2$. Now, we define

$$\beta = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} J(\gamma(s)),$$

where $\Gamma = \{\gamma \in C([0,1]\,,\, H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega))\,:\,\gamma(0) = w_1 \quad \text{and } \gamma(1) = w_1 + l_0\eta w_\varepsilon\}$. Define a path $\gamma(s) = w_1 + sl_0\eta w_\varepsilon$ for $s\in[0\,,\,1]$, and we have $\gamma(0)\in W_1$, $\gamma(1)\in W_2$. Then there exists $s_0 \in (0\,,\,1)$ such that $\gamma(s_0) \in \mathcal{N}^-$, and we have $\beta > m_-$. Therefore, by Lemma 3.2, we get

$$m_- \leq \beta < m_J + \frac{\alpha}{2N}\Big(\kappa_\alpha S(\alpha, N)\Big)^{\frac{N}{\alpha}}.$$

Analogously to the proof of Proposition 3.1, one can show that Ekeland's variational principle gives a sequence $\{w_n\} \in \mathcal{N}^-$ which satisfies

$$J(w_n) \to m_- \quad \text{and} \quad J'(w_n) \to 0 \quad \text{as } n\to\infty.$$

Since $m_- < m_J + \frac{\alpha}{2N}\Big(\kappa_\alpha S(\alpha,N)\Big)^{\frac{N}{\alpha}}$, by Proposition 3.2 and Lemma 3.1, there exists a subsequence $\{w_n\}$ and w_2 such that

$$w_n \to w_2 \quad \text{strongly in } \ H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega),$$

$w_2 \in \mathcal{N}^-$ and $J(w_2) = m_-$.

Since $J(w_2) = J(|w_2|)$ and $|w_2| \in \mathcal{N}^-$, we can always take $w_2 \geq 0$. By the maximum principle [11], we get $w_2 > 0$ in $H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)$. Hence, $u_2(x) = w_2(\cdot, 0) \in H_0^{\frac{\alpha}{2}}(\Omega)$ is a positive solution of problem (1.1). The proof is completed. □

Proof of Theorem 1.1. By Theorem 3.2 and Proposition 3.3, the equation (2.1) has two positive solutions w_1 and w_2 such that $w_1 \in \mathcal{N}^+$ and $w_2 \in \mathcal{N}^+$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$. This implies that problem (1.1) has at least two positive solutions $u_1(x) = w_1(x,0)$ and $u_2(x) = w_2(x,0)$.

4. Concentration Behavior

In this section, we give the proof of Theorem 1.2.

For every $\mu > 0$, we define

$$J_\mu(w) = \frac{\kappa_\alpha}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dxdy - \frac{\mu}{2^*_\alpha}\int_{\Omega\times\{0\}} |w|^{2^*_\alpha}dx;$$

$$\mathcal{O}_\mu=\{w\in H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega): w\not\equiv 0 \text{ and } \langle J'_\mu(w)\,,\,w\rangle=0\}.$$

We have the following lemmas.

Lemma 4.1. *For every $w\in\mathcal{N}^-$, there is a unique $t(w)>0$ such that $t(w)w\in\mathcal{O}_1$ and*

$$1-\lambda|f|_{L^\infty}\Big(\frac{2^*_\alpha-q}{S_0(2-q)}\Big)^{\frac{2^*_\alpha-q}{2^*_\alpha-2}}\le t^{2^*_\alpha-2}(w)\le 1+\lambda|f|_{L^\infty}\Big(\frac{2^*_\alpha-q}{S_0(2-q)}\Big)^{\frac{2^*_\alpha-q}{2^*_\alpha-2}}, \quad (4.1)$$

where $S_0=\kappa_\alpha S(\alpha,N)$.

Proof. For each $w\in\mathcal{N}^-$, we have

$$\kappa_\alpha\int\limits_{\mathcal{C}_\Omega}y^{1-\alpha}|\nabla w|^2dxdy-\lambda\int\limits_{\Omega\times\{0\}}f(x)|w|^q dx-\int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha}dx=0 \quad (4.2)$$

and

$$0<(2-q)\kappa_\alpha\int\limits_{\mathcal{C}_\Omega}y^{1-\alpha}|\nabla w|^2dxdy<(2^*_\alpha-q)\int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha}dx. \quad (4.3)$$

Thus, from (4.3), the functional

$$J_1(tw)=t^2\,\frac{\kappa_\alpha}{2}\int\limits_{\mathcal{C}_\Omega}y^{1-\alpha}|\nabla w|^2dxdy-\frac{t^{2^*_\alpha}}{2^*_\alpha}\int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha}dx$$

with respect to t is initially increasing and eventually decreasing and with a single turning point $t(w)$ such that $t(w)w\in\mathcal{O}_1$. So

$$t^2(w)\kappa_\alpha\int\limits_{\mathcal{C}_\Omega}y^{1-\alpha}|\nabla w|^2dxdy=t^{2^*_\alpha}(w)\int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha}dx. \quad (4.4)$$

Then, from (4.2), (4.4) and The Hölder inequality

$$\begin{aligned}1-\lambda|f|_{L^\infty}\,|w|^{-(2^*_\alpha-q)}_{L^{2^*_\alpha}}\le t^{2^*_\alpha-2}(w)&=\frac{\kappa_\alpha\int\limits_{\mathcal{C}_\Omega}y^{1-\alpha}|\nabla w|^2dxdy}{\int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha}dx}\\&=1+\frac{\lambda\int\limits_{\Omega\times\{0\}}f(x)|w|^q dx}{\int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha}dx}\\&\le 1+\lambda|f|_{L^\infty}\,|w|^{-(2^*_\alpha-q)}_{L^{2^*_\alpha}}\end{aligned} \quad (4.5)$$

On the other hand, by (2.2) and (4.3), we get

$$\begin{aligned}\int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha}dx&>\frac{2-q}{2^*_\alpha-q}\kappa_\alpha\int\limits_{\mathcal{C}_\Omega}y^{1-\alpha}|\nabla w|^2dxdy\\&\ge\frac{2-q}{2^*_\alpha-q}\kappa_\alpha S(\alpha,N)\Big(\int\limits_{\Omega\times\{0\}}|w|^{2^*_\alpha}dx\Big)^{\frac{2}{2^*_\alpha}},\end{aligned}$$

that is

$$|w|_{L^{2^*_\alpha}}>\Big(\frac{(2-q)\kappa_\alpha S(\alpha,N)}{2^*_\alpha-q}\Big)^{\frac{1}{2^*_\alpha-2}}. \quad (4.6)$$

Hence, from (4.6) and (4.5), we obtain (4.1). This completes the proof. $\square$

Remark 4.1. *From* (4.1)*, it is easy to see that* $t(w) \to 1$ *as* $\lambda \to 0$*.*

Proof the Theorem 1.2. Suppose that $\{\lambda_n\}$ is a sequence of positive number such that $\lambda_n \to 0$ as $n \to +\infty$. Let $w_n^{(1)} = w_{1,n} \in \mathcal{N}^+$ and $w_n^{(2)} = w_{2,n} \in \mathcal{N}^-$ are position solutions of equation (2.1) corresponding to $\lambda = \lambda_n$. We have two following results.

(i) By Remark 3.1, for every $w_n^{(1)} \in \mathcal{N}^+$, we can conclude that $\|w_n^{(1)}\|_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \to 0$ as $n \to \infty$.
(ii) By Lemma 4.1 and Remark 4.1, for every $w_n^{(2)} \in \mathcal{N}^-$, there is a unique $t(w_n^{(2)}) > 0$ such that $t(w_n^{(2)})\, w_n^{(2)} \in \mathcal{O}_1$, and $t(w_n^{(2)}) \to 1$ as $n \to \infty$.

For case (ii). For each $w_n^{(2)} \in \mathcal{N}^-$, let

$$f(t) = J_\mu(tw_n^{(2)}) = t^2 \frac{\kappa_\alpha}{2} \int\limits_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w_n^{(2)}|^2 dxdy - t^{2^*_\alpha} \frac{\mu}{2^*_\alpha} \int\limits_{\Omega\times\{0\}} |w_n^{(2)}|^{2^*_\alpha} dx.$$

Since $f(t) \to -\infty$ as $s \to \infty$, $\sup\limits_{t\geq 0} f(t)$ is achieved at some $\widetilde{t} > 0$ with $f'(\widetilde{t}) = 0$, which is

$$f'(\widetilde{t}) = \widetilde{t}\Big(\|w_n^{(2)}\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \widetilde{t}^{2^*_\alpha - 2} \mu \int\limits_{\Omega\times\{0\}} |w_n^{(2)}|^{2^*_\alpha} dx \Big) = 0.$$

Let

$$\widetilde{t} = \Big(\frac{\|w_n^{(2)}\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}{\mu \int\limits_{\Omega\times\{0\}} |w_n^{(2)}|^{2^*_\alpha} dx} \Big)^{\frac{1}{2^*_\alpha - 2}}.$$

Then $\widetilde{t}\, w_n^{(2)} \in \mathcal{O}_\mu$ and

$$\sup_{t\geq 0} J_\mu(tw_n^{(2)}) = J_\mu(\widetilde{t}\, w_n^{(2)}) = \frac{\alpha}{2N} \Big(\frac{\|w_n^{(2)}\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}}{\mu \int\limits_{\Omega\times\{0\}} |w_n^{(2)}|^{2^*_\alpha} dx} \Big)^{\frac{N-\alpha}{2}}.$$

On the other hand, by Hölder inequality and Young inequality, for $\mu \in (0\,, 1)$, we have

$$\begin{aligned}
\int\limits_{\Omega\times\{0\}} f(x) |\widetilde{t} w_n^{(2)}|^q dx &\leq |f|_{L^\infty} \Big(\int\limits_{\Omega\times\{0\}} |\widetilde{t} w_n^{(2)}|^{2^*_\alpha} dx \Big)^{\frac{q}{2^*_\alpha}} \\
&\leq |f|_{L^\infty} \Big(\kappa_\alpha\, S(\alpha, N) \Big)^{-\frac{q}{2}} \widetilde{t}^q \, \|w_n^{(2)}\|^q_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \\
&\leq \frac{2-q}{2} \Big(|f|_{L^\infty} (\kappa_\alpha\, S(\alpha, N)\, \mu)^{-\frac{q}{2}} \Big)^{\frac{2}{2-q}} + \mu \frac{q}{2} \Big(\widetilde{t}^q \, \|w_n^{(2)}\|^q_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} \Big)^{\frac{2}{q}} \\
&= \frac{2-q}{2} \mu^{\frac{-q}{2-q}} \Big(|f|_{L^\infty} (\kappa_\alpha\, S(\alpha, N))^{-\frac{q}{2}} \Big)^{\frac{2}{2-q}} + \frac{\mu q}{2} \|\widetilde{t} w_n^{(2)}\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)}.
\end{aligned}$$

Then we get

$$\begin{aligned}
J(\tilde{t}\,w_n^{(2)}) &= \frac{1}{2}\|\tilde{t}\,w_n^{(2)}\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \frac{\lambda}{q}\int\limits_{\Omega\times\{0\}} f(x)|\tilde{t}\,w_n^{(2)}|^q dx - \frac{1}{2^*_\alpha}\int\limits_{\Omega\times\{0\}} |\tilde{t}\,w_n^{(2)}|^{2^*_\alpha} dx \\
&\geq \frac{1-\lambda\mu}{2}\|\tilde{t}\,w_n^{(2)}\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \frac{\lambda(2-q)}{2q}\mu^{\frac{-q}{2-q}}\Big(|f|_{L^\infty}(\kappa_\alpha\, S(\alpha,N))^{-\frac{q}{2}}\Big)^{\frac{2}{2-q}} \\
&\quad - \frac{1}{2^*_\alpha}\int\limits_{\Omega\times\{0\}} |\tilde{t}\,w_n^{(2)}|^{2^*_\alpha} dx \\
&= (1-\lambda\mu)\Big(\frac{1}{2}\|\tilde{t}\,w_n^{(2)}\|^2_{H^{\frac{\alpha}{2}}_{0,L}(\mathcal{C}_\Omega)} - \frac{\frac{1}{1-\lambda\mu}}{2^*_\alpha}\int\limits_{\Omega\times\{0\}} |\tilde{t}\,w_n^{(2)}|^{2^*_\alpha} dx\Big) \\
&\quad - \frac{\lambda(2-q)}{2q}\mu^{\frac{-q}{2-q}}\Big(|f|_{L^\infty}(\kappa_\alpha\, S(\alpha,N))^{-\frac{q}{2}}\Big)^{\frac{2}{2-q}} \\
&= (1-\lambda\mu)J_{\frac{1}{1-\lambda\mu}}(\tilde{t}\,w_n^{(2)}) - \frac{\lambda(2-q)}{2q}\mu^{\frac{-q}{2-q}}\Big(|f|_{L^\infty}(\kappa_\alpha\, S(\alpha,N))^{-\frac{q}{2}}\Big)^{\frac{2}{2-q}} \\
&= (1-\lambda\mu)^{\frac{N-\alpha+2}{2}}J_1(\tilde{t}\,w_n^{(2)}) - \frac{\lambda(2-q)}{2q}\mu^{\frac{-q}{2-q}}\Big(|f|_{L^\infty}(\kappa_\alpha\, S(\alpha,N))^{-\frac{q}{2}}\Big)^{\frac{2}{2-q}}.
\end{aligned} \tag{4.7}$$

Therefore, corresponding to $\lambda = \lambda_n$, from (4.7), Remark 4.1 and the fact

$$J(w_n^{(2)}) < m_J + \frac{\alpha}{2N}\Big(\kappa_\alpha S(\alpha,N)\Big)^{\frac{N}{\alpha}},$$

we obtain

$$\begin{aligned}
&J_1(\tilde{t}\,w_n^{(2)}) \\
&\leq \Big(\frac{1}{1-\lambda_n\mu}\Big)^{\frac{N-\alpha+2}{2}}\Big[J(\tilde{t}\,w_n^{(2)}) + \frac{\lambda_n(2-q)}{2q}\mu^{\frac{-q}{2-q}}\Big(|f|_{L^\infty}(\kappa_\alpha\, S(\alpha,N))^{-\frac{q}{2}}\Big)^{\frac{2}{2-q}}\Big] \\
&< \Big(\frac{1}{1-\lambda_n\mu}\Big)^{\frac{N-\alpha+2}{2}}\Big[m_J + \frac{\alpha}{2N}(\kappa_\alpha S(\alpha,N))^{\frac{N}{\alpha}} \\
&+ \frac{\lambda_n(2-q)}{2q}\mu^{\frac{-q}{2-q}}\Big(|f|_{L^\infty}(\kappa_\alpha\, S(\alpha,N))^{-\frac{q}{2}}\Big)^{\frac{2}{2-q}}\Big].
\end{aligned}$$

Since $m_J \to 0$, $\tilde{t} \to 1$ as $n \to \infty$, it is easy to see that

$$\limsup_{n\to\infty} J_1(w_n^{(2)}) \leq \frac{\alpha}{2N}(\kappa_\alpha S(\alpha,N))^{\frac{N}{\alpha}}.$$

This tell us

$$\lim_{n\to\infty} J_1(w_n^{(2)}) = \frac{\alpha}{2N}(\kappa_\alpha S(\alpha,N))^{\frac{N}{\alpha}}.$$

We can conclude that $\{w_n^{(2)}\}$ is a minimizing sequence for J_1 in $\mathcal{O}_1$. Then

$$\kappa_\alpha\int\limits_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w_n^{(2)}|^2 dxdy - \int\limits_{\Omega\times\{0\}} |w_n^{(2)}|^{2^*_\alpha} dx \to 0$$

and

$$J_1(w_n^{(2)}) \to \frac{\alpha}{2N}(\kappa_\alpha S(\alpha,N))^{\frac{N}{\alpha}}$$

as $n \to \infty$. This implies that $\{w_n^{(2)}\}$ is a $(PS)_c$-sequence for J_1 at level $c = \frac{\alpha}{2N}(\kappa_\alpha S(\alpha, N))^{\frac{N}{\alpha}}$. Clearly, $\{w_n^{(2)}\}$ is bounded, and then there exists a subsequence $\{w_n^{(2)}\}$ and $w_0 \in H_{0,L}^{\frac{\alpha}{2}}(\mathcal{C}_\Omega)$ such that

$$w_n^{(2)} \rightharpoonup w_0 \quad \text{weakly in } H_{0,L}^{\frac{\alpha}{2}}(\mathcal{C}_\Omega).$$

Since Ω is bounded, we have $w_0 = 0$. Moreover, by the concentration-compactness principle (see Theorem 6 of [10]), there exist two sequence $\{x_n\} \subset \Omega$, $\{R_n\} \subset \mathbb{R}^+$ such that $R_n \to \infty$ as $n \to \infty$ and

$$\|\mathrm{tr}_\Omega w_n^{(2)} - R_n^{\frac{N-\alpha}{2}} u(R_n(x - x_n))\|_{H_0^{\frac{\alpha}{2}}(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

This completes the proof of Theorem 1.2.

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